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BENDING VIBRATIONS OF BEAMS COUPLED BY SEVERAL DOUBLE SPRING-MASS SYSTEMS

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1. INTRODUCTION

The recent study [1], published by İnceoğlu and Gürgöze on the problem of the natural longitudinal vibrations of two rods coupled by several double spring-mass systems, constitutes the motivation behind this note. It is concerned with the transverse vibrations of a system consisting of two clamped–free Bernoulli–Euler beams carrying tip masses to which several double spring-mass systems are attached across the span. The main purpose of the study is to derive a formulation for obtaining the natural frequencies of the system described by using Green's function method.

The special case of a symmetrical system can be a model of a suspension bridge tower for studying its bending vibrations, among other applications.

2. THEORY

The problem to be dealt with in the present study is the natural vibration problem of the system shown in Figure 1 i.e., a laterally vibrating system consisting of two clamped–free beams carrying tip masses to which several double spring-mass systems (secondary system: ss) are attached across the span.

However, to aid the explanation on the one hand and to clarify the physics of the problem on the other, Green's function method will be first applied to one ss, and then it will be generalized to n ss.

2.1. THE CASE OF ONE SS, $n = 1$

The combined system, which has already been studied in reference [2] and is to be investigated initially, consists of two clamped–free beams carrying tip masses to which a double spring-mass system is attached in span, as seen in Figure 2. L_i, m_i, $\eta_i L_i$ and $E_i I_i$ denote the length, the mass per unit length, the location of the spring attachment point and the lateral rigidity of the ith beam, respectively ($i = 1, 2$). The secondary system consists of two springs of stiffnesses k_1, k_2 and the mass M_s. Further, the bending vibration displacements of the first and second beams are denoted as $w_1(x, t)$ and $w_2(x, t)$, respectively, and $z(t)$ represents the displacement of the mass M_s.

Assuming the spring forces, which are generated by the secondary system, as singular effects for both of the beams, the bending vibration equations of the system can be written in

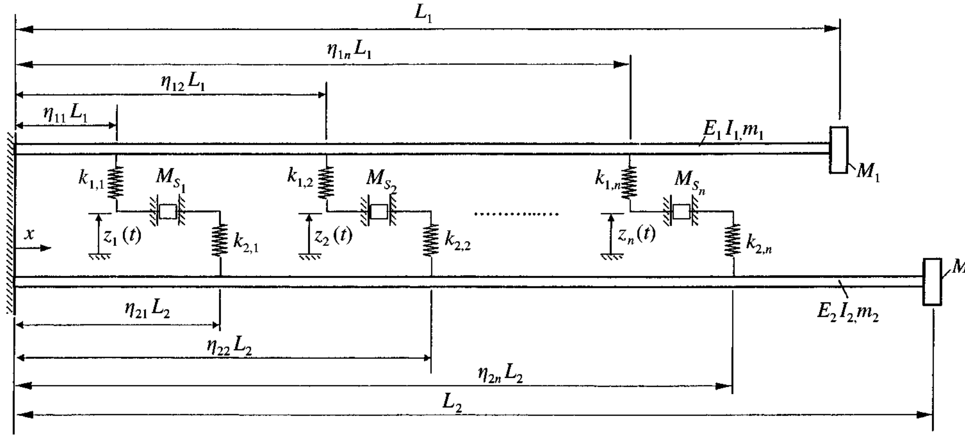

Figure 1. Two clamped–free laterally vibrating beams carrying tip masses to which several double spring-mass systems are attached across the span.

the following form:

$$E_iI_i\frac{\partial^4}{\partial x^4}w_i(x,t)+m_i\frac{\partial^2}{\partial t^2}w_i(x,t)=k_i[z(t)-w_i(\eta_iL_i,t)]\delta(x-\eta_iL_i),\quad i=1,2. \tag{1}$$

The motion of the secondary mass is governed by

$$M_s\ddot{z}(t)=-k_1[z(t)-w_1(\eta_1L_1,t)]+k_2[w_2(\eta_2L_2,t)-z(t)], \tag{2}$$

where $\delta(\cdot)$ denotes the Dirac delta function and dots and primes denote partial derivatives with respect to time t and position co-ordinate x respectively.

Using separation of variables according to

$$w_i(x,t)=W_i(x)\cos\omega t,\quad i=1,2,$$

$$z(t)=Z\cos\omega t, \tag{3}$$

where $W_i(x)$ and Z are the corresponding amplitude functions and ω is the unknown eigenfrequency of the combined system, and putting them into equations (1) and (2),

$$W_1^{IV}(x)-\beta^4W_1(x)=\frac{k_1}{E_1I_1}\left\{\frac{k_2[W_2(h_2)-W_1(h_1)]+M_s\omega^2W_1(h_1)}{k_1+k_2-M_s\omega^2}\right\}\delta(x-h_1),$$

$$W_2^{IV}(x)-\mu^4\beta^4W_2(x)=\frac{k_2}{E_2I_2}\left\{\frac{k_1[W_1(h_1)-W_2(h_2)]+M_s\omega^2W_2(h_2)}{k_1+k_2-M_s\omega^2}\right\}\delta(x-h_2), \tag{4}$$

are obtained, where

$$h_i=\eta_iL_i,\qquad \beta^4=\frac{m_1\omega^2}{E_1I_1},\qquad \mu^4=\frac{\alpha_m}{\chi},\qquad \alpha_m=\frac{m_2}{m_1},\qquad \chi=\frac{E_2I_2}{E_1I_1},\quad i=1,2 \tag{5}$$

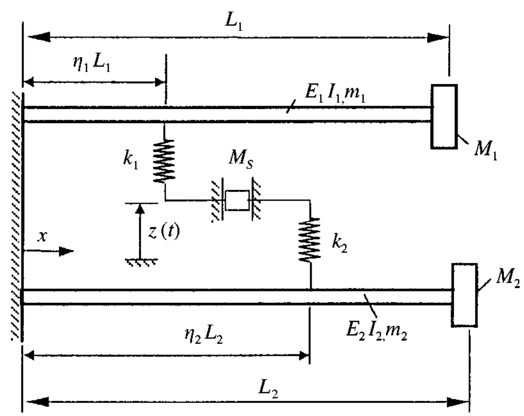

Figure 2. Two clamped–free laterally vibrating beams carrying tip masses to which a double spring-mass system is attached across the span.

Using the non-dimensional parameters

$$\xi_i = \frac{x}{L_i}, \qquad \bar{W}_i = \frac{W_i}{L_i}, \qquad \bar{\beta} = \beta L_1, \qquad \alpha_{k_i} = \frac{k_i}{E_i I_i / L_i^3}, \qquad \alpha_k = \frac{k_2}{k_1}, \qquad \alpha_M = \frac{M_s}{m_1 L_1}$$

$$\alpha_L = \frac{L_2}{L_1}, \qquad \bar{\delta} = \mu \alpha_L, \qquad (\cdot)' \hat{=} \frac{\partial}{\partial \xi}, \quad i = 1, 2 \tag{6}$$

the above equations can be reformulated as

$$\bar{W}_1^{\mathrm{IV}}(\xi_1) - \bar{\beta}^4 \bar{W}_1(\xi_1) = \left\{ \frac{\alpha_k \alpha_{k_1} [\alpha_L \bar{W}_2(\eta_2) - \bar{W}_1(\eta_1)] + \alpha_M \bar{\beta}^4 \bar{W}_1(\eta_1)}{1 + \alpha_k - (\alpha_M / \alpha_{k_1}) \bar{\beta}^4} \right\} \delta(\xi_1 - \eta_1),$$

$$\bar{W}_2^{\mathrm{IV}}(\xi_2) - \bar{\delta}^4 \bar{\beta}^4 \bar{W}_2(\xi_2) = \left\{ \frac{\alpha_{k_2} [(1/\alpha_L) \bar{W}_1(\eta_1) - \bar{W}_2(\eta_2)] + (\alpha_{k_2}/\alpha_{k_1}) \alpha_M \bar{\beta}^4 \bar{W}_2(\eta_2)}{1 + \alpha_k - (\alpha_M / \alpha_{k_1}) \bar{\beta}^4} \right\} \delta(\xi_2 - \eta_2). \tag{7}$$

For the solution of the above differential equations, Green's function method will be employed. For the sake of completeness, the derivation of the corresponding Green's function is given in Appendix A. Therefore, via an analogy with (A3), i.e., using $\xi_1, \eta_1, \bar{\beta}, G_1(\xi_1, \eta_1)$, 1 for the first beam and $\xi_2, \eta_2, \bar{\delta}\,\bar{\beta}, G_2(\xi_2, \eta_2)$, 1 for the second beam, instead of $x, \xi, \beta, G(x, \xi), L$ respectively, Green's functions, which correspond to the combined system, can be written as follows:

$$G_1(\xi_1, \eta_1) = \frac{1}{2\bar{\beta}^3} \{ \Phi_4(\bar{\beta}(\xi_1 - \eta_1)) \mathrm{H}(\xi_1 - \eta_1) + \bar{\beta} G''(0, \eta_1) \Phi_3(\bar{\beta}\xi_1) + G'''(0, \eta_1) \Phi_4(\bar{\beta}\xi_1) \},$$

$$G_2(\xi_2, \eta_2) = \frac{1}{2(\bar{\delta}\bar{\beta})^3} \{ \Phi_4(\bar{\delta}\bar{\beta}(\xi_2 - \eta_2)) \mathrm{H}(\xi_2 - \eta_2) + \bar{\delta}\bar{\beta} G''(0, \eta_2) \Phi_3(\bar{\delta}\bar{\beta}\xi_2)$$
$$+ G'''(0, \eta_2) \Phi_4(\bar{\delta}\bar{\beta}\xi_2) \}, \tag{8}$$

where

$$\Phi_1(x) = \cosh x + \cos x, \quad \Phi_2(x) = \sinh x + \sin x,$$

$$\Phi_3(x) = \cosh x - \cos x, \quad \Phi_4(x) = \sinh x - \sin x. \tag{9}$$

Here H(·) denotes the Heaviside unit step function. Now, the displacements of the points $\xi_1 = \eta_1$ and $\xi_2 = \eta_2$ can be given in the form

$$\bar{W}_1(\eta_1) = G_1(\eta_1, \eta_1)\left\{\frac{\alpha_k\alpha_{k_1}[\alpha_L\bar{W}_2(\eta_2) - \bar{W}_1(\eta_1)] + \alpha_M\bar{\beta}^4\bar{W}_1(\eta_1)}{[1 + \alpha_k - (\alpha_M/\alpha_{k_1})\bar{\beta}^4]}\right\},$$

$$\bar{W}_2(\eta_2) = G_2(\eta_2, \eta_2)\left\{\frac{\alpha_{k_2}[(1/\alpha_L)\bar{W}_1(\eta_1) - \bar{W}_2(\eta_2)] + (\alpha_{k_2}/\alpha_{k_1})\alpha_M\bar{\beta}^4\bar{W}_2(\eta_2)}{[1 + \alpha_k - (\alpha_M/\alpha_{k_1})\bar{\beta}^4]}\right\}. \tag{10}$$

These equations represent a set of two homogeneous equations for the solution of the unknowns $\bar{W}_1(\eta_1)$ and $\bar{W}_2(\eta_2)$. A non-trivial solution exists when the determinant of the coefficient matrix vanishes. This condition in turn leads to the following frequency equation:

$$\begin{vmatrix} (1 - G_1C_{11}) & -G_1C_{12} \\ -G_2C_{21} & (1 - G_2C_{22}) \end{vmatrix} = 0, \tag{11}$$

where,

$$C_{11} = \frac{[\alpha_M\bar{\beta}^4 - \alpha_k\alpha_{k_1}]}{[1 + \alpha_k - (\alpha_M/\alpha_{k_1})\bar{\beta}^4]}, \qquad C_{12} = \frac{\alpha_k\alpha_{k_1}\alpha_L}{[1 + \alpha_k - (\alpha_M/\alpha_{k_1})\bar{\beta}^4]},$$

$$C_{21} = \frac{\alpha_{k_2}/\alpha_L}{[1 + \alpha_k - (\alpha_M/\alpha_{k_1})\bar{\beta}^4]}, \qquad C_{22} = \frac{(\alpha_{k_2}/\alpha_{k_1})\alpha_M\bar{\beta}^4 - \alpha_{k_2}}{[1 + \alpha_k - (\alpha_M/\alpha_{k_1})\bar{\beta}^4]},$$

$$G_i \mathrel{\hat{=}} G_i(\eta_i, \eta_i), \quad i = 1, 2. \tag{12}$$

The solution of equation (11) yields the desired non-dimensional frequency parameters $\bar{\beta}$ *of the combined system.*

2.2. GENERALIZATION FOR THE CASE OF SEVERAL SS'S $n = n$

Consider a system of two beams that are carrying tip masses and coupled by n ss's in such a way that n points of the first beam of co-ordinates $\eta_{11}, \eta_{12}, \ldots, \eta_{1n}$ are connected to n points of co-ordinates $\eta_{21}, \eta_{22}, \ldots, \eta_{2n}$ belonging to the second beam, by using springs of stiffnesses $k_{1,1}, k_{2,1}, k_{1,2}, k_{2,2}, \ldots, k_{1,n}, k_{2,n}$ and the masses $M_{s_1}, M_{s_2}, \ldots, M_{s_n}$ made up of ss's, as shown in Figure 1.

Considering equations (7) representing the governing differential equations of the combined system having one ss, i.e. $n = 1$, can be reformulated for the case of several ss's, i.e. $n = n$, as below:

$$\bar{W}_1^{\mathrm{IV}}(\xi_1) - \bar{\beta}^4\bar{W}_1(\xi_1) = \sum_{j=1}^{n}\left\{\frac{\alpha_{k_j}\alpha_{k_{1,j}}[\alpha_L\bar{W}_2(\eta_{2j}) - \bar{W}_1(\eta_{1j})] + \alpha_{M_j}\bar{\beta}^4\bar{W}_1(\eta_{1j})}{[1 + \alpha_{k_j} - (\alpha_{M_j}/\alpha_{k_{1,j}})\bar{\beta}^4]}\right\}\delta(\xi_1 - \eta_{1j}),$$

$$\bar{W}_2^{\mathrm{IV}}(\xi_2) - \bar{\delta}^4 \bar{\beta}^4 \bar{W}_2(\xi_2)$$
$$= \sum_{j=1}^{n} \left\{ \frac{\alpha_{k_{2,j}}[(1/\alpha_L)\bar{W}_1(\eta_{1j}) - \bar{W}_2(\eta_{2j})] + (\alpha_{k_{2,j}}/\alpha_{k_{1,j}})\alpha_{M_j}\bar{\beta}^4 \bar{W}_2(\eta_{2j})}{[1 + \alpha_{k_j} - (\alpha_{M_j}/\alpha_{k_{1,j}})\bar{\beta}^4]} \right\} \delta(\xi_2 - \eta_{2j}). \quad (13)$$

Similarly, for this case, equations (10) can be rearranged as

$$\bar{W}_1(\xi_1) = \sum_{j=1}^{n} G_1(\xi_1, \eta_{1j}) \left\{ \frac{\alpha_{k_j}\alpha_{k_{1,j}}[\alpha_L \bar{W}_2(\eta_{2j}) - \bar{W}_1(\eta_{1j})] + \alpha_{M_j}\bar{\beta}^4 \bar{W}_1(\eta_{1j})}{[1 + \alpha_{k_j} - (\alpha_{M_j}/\alpha_{k_{1,j}})\bar{\beta}^4]} \right\},$$

$$\bar{W}_2(\xi_2) = \sum_{j=1}^{n} G_2(\xi_2, \eta_{2j}) \left\{ \frac{\alpha_{k_{2,j}}[(1/\alpha_L)\bar{W}_1(\eta_{1j}) - \bar{W}_2(\eta_{2j})] + (\alpha_{k_{2,j}}/\alpha_{k_{1,j}})\alpha_{M_j}\bar{\beta}^4 \bar{W}_2(\eta_{2j})}{[1 + \alpha_{k_j} - (\alpha_{M_j}/\alpha_{k_{1,j}})\bar{\beta}^4]} \right\}. \quad (14)$$

For simplicity, these equations can be written in the following form after some arrangements:

$$\bar{W}_1(\xi_1) = \sum_{j=1}^{n} [(C_{1j}^{(1)} \bar{W}_1(\eta_{1j}) + C_{2j}^{(1)} \bar{W}_2(\eta_{2j})) G_1(\xi_1, \eta_{1j})],$$

$$\bar{W}_2(\xi_2) = \sum_{j=1}^{n} [(C_{1j}^{(2)} \bar{W}_1(\eta_{1j}) + C_{2j}^{(2)} \bar{W}_2(\eta_{2j})) G_2(\xi_2, \eta_{2j})], \quad (15)$$

where the following abbreviations are introduced:

$$C_{1j}^{(1)} = \frac{[\alpha_{M_j}\bar{\beta}^4 - \alpha_{k_j}\alpha_{k_{1,j}}]}{[1 + \alpha_{k_j} - (\alpha_{M_j}/\alpha_{k_{1,j}})\bar{\beta}^4]}, \qquad C_{2j}^{(1)} = \frac{\alpha_{k_j}\alpha_{k_{1,j}}\alpha_L}{[1 + \alpha_{k_j} - (\alpha_{M_j}/\alpha_{k_{1,j}})\bar{\beta}^4]},$$

$$C_{1j}^{(2)} = \frac{\alpha_{k_{2,j}}/\alpha_L}{[1 + \alpha_{k_j} - (\alpha_{M_j}/\alpha_{k_{1,j}})\bar{\beta}^4]}, \qquad C_{2j}^{(2)} = \frac{(\alpha_{k_{2,j}}/\alpha_{k_{1,j}})\alpha_{M_j}\bar{\beta}^4 - \alpha_{k_{2,j}}}{[1 + \alpha_{k_j} - (\alpha_{M_j}/\alpha_{k_{1,j}})\bar{\beta}^4]},$$

$$\alpha_{k_j} = \frac{k_{2,j}}{k_{1,j}}, \qquad \alpha_{M_j} = \frac{M_{s_j}}{m_1 L_1}, \qquad \alpha_{k_{1,j}} = \frac{k_{1,j}}{E_1 I_1/L_1^3}, \qquad \alpha_{k_{2,j}} = \frac{k_{2,j}}{E_2 I_2/L_2^3},$$

$$\alpha_L = \frac{L_2}{L_1}, \quad j = 1, 2, \ldots, n. \quad (16)$$

Equations (15) represent displacement fields on the axes ξ_1 and ξ_2. In order to find the displacements at all the attachment points along the axes,

$$\xi_1 \to \eta_{11}, \eta_{12}, \ldots, \eta_{1n},$$

$$\xi_2 \to \eta_{21}, \eta_{22}, \ldots, \eta_{2n}, \quad (17)$$

have to be substituted in equations (15). Thus, $2n$ equations are obtained for the $2n$ unknowns $\bar{W}_1(\eta_{1j})$ and $\bar{W}_2(\eta_{2j})$, $j = 1, 2, \ldots, n$.

Using matrix notation,

$$\mathbf{Ax} = \mathbf{0} \quad (18)$$

can be written, where

$$\mathbf{x}^{\mathrm{T}} = \{\bar{W}_1(\eta_{11}), \bar{W}_1(\eta_{12}), \ldots, \bar{W}_1(\eta_{1n}) | \bar{W}_2(\eta_{21}), \bar{W}_2(\eta_{22}), \ldots, \bar{W}_2(\eta_{2n})\},$$

$$\mathbf{A} = \left[\begin{array}{c|c} \mathbf{I}_n - \mathbf{C}_1^{(1)} & \mathbf{C}_2^{(1)} \\ \hline \mathbf{C}_1^{(2)} & \mathbf{I}_n - \mathbf{C}_2^{(2)} \end{array}\right]. \tag{19}$$

$\mathbf{I}_n : n \times n$ unit matrix,

$$[\mathbf{C}_1^{(1)}]_{ij} = C_{1j}^{(1)} G_{ij}^{(1)}, \qquad [\mathbf{C}_2^{(1)}]_{ij} = C_{2j}^{(1)} G_{ij}^{(1)}, \qquad [\mathbf{C}_1^{(2)}]_{ij} = C_{1j}^{(2)} G_{ij}^{(2)}, \qquad [\mathbf{C}_2^{(2)}]_{ij} = C_{2j}^{(2)} G_{ij}^{(2)},$$

$$G_{ij}^{(m)} \triangleq G_m(\eta_{mi}, \eta_{mj}), \quad i, j = 1, 2, \ldots, n, \ m = 1, 2. \tag{20}$$

Here, $[\cdot]_{ij}$ denotes an element of the corresponding matrix located at the ith row and the jth column.

A non-trivial solution exists when the determinant of the coefficient matrix **A** vanishes. Thus, the following frequency equation can be obtained:

$$\det(\mathbf{A}) = 0. \tag{21}$$

The solution of equation (21) yields the non-dimensional frequency parameters $\bar{\beta}$ of the system in Figure 1.

3. NUMERICAL RESULTS

This section is devoted to the numerical evaluations of the formulae established in the preceding sections. As an example, the $n = 1$ case, i.e., one ss case is considered. Normally, the classical approach of deriving the frequency equation based on the boundary value problem formulation is fairly complicated even for the case of one ss which is presented in

TABLE 1

The first 10 dimensionless eigenfrequency parameters $\bar{\beta}$ of the system in Figure 2, i.e. $n = 1$†

From equation (21)	From equation (B1)
1·07026286	1·07026286
1·55878983	1·55878983
3·30959152	3·30959152
6·33425479	6·33425479
6·81983036	6·81983036
7·46938324	7·46938324
7·92595594	7·92595594
10·66673582	10·66673582
10·74697127	10·74697127
13·40231846	13·40231846

† $\eta_1 = \eta_2 = 0{\cdot}5$, $\alpha_{M_1} = \alpha_{M_2} = 2$, $\alpha_{k_1} = \alpha_{k_2} = 1000$ with all other dimensionless parameters set to one.

reference [2]. Practically, as the number of the ss exceeds one, the solution of the problem becomes nearly impossible and extremely tedious. Since the solution of the problem for a number of the ss's n, does not exist in the literature to the best of our knowledge, the only way to prove the validity of the present formulation is to compare the present results with obtained using a classical approach for one ss, i.e. $n = 1$, in Figure 2.

In Table 1, the first 10 dimensionless eigenfrequency parameters $\bar{\beta}$ of the described system are given for the numerical values, taken from reference [2], $\eta_1 = \eta_2 = 0{\cdot}5$, $\alpha_{M_1} = \alpha_{M_2} = 2$, $\alpha_{k_1} = \alpha_{k_2} = 1000$ with all other dimensionless parameters set to one. The values in the first column are the roots of equation (21) derived via Green's function method, whereas those in the second column are values from the solution of equation (B1). It is seen clearly that the values in the columns are identical and justify the lengthy and complicated expressions obtained by the application of Green's function method.

4. CONCLUSION

This study is concerned with the bending vibrations of a combined system consisting of two clamped–free beams carrying tip masses to which several double spring-mass systems are attached across the span. Using Green's function method, the frequency equation of the system with n ss's is established. Then in order to prove the validity of the expressions derived, for a special system with $n = 1$, the results are compared with those obtained on the basis of a boundary value problem formulation. The two results are in excellent agreement which clearly indicates the validity of the formulae obtained via Green's function method.

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APPENDIX A

The corresponding Green's function for the clamped–free beam carrying a tip mass, is the solution of the differential equation

$$\frac{d^4 G(x, \xi)}{dx^4} - \beta^4 G(x, \xi) = \delta(x - \xi) \tag{A1}$$

subject to the following boundary conditions:

$$\begin{aligned} &G(0, \xi) = G'(0, \xi) = 0 \quad \text{at the clamped end,} \\ &G''(L, \xi) = 0 \\ &G'''(L, \xi) + A\beta^4 G(L, \xi) = 0 \quad \text{at the free end.} \end{aligned} \tag{A2}$$

where A denotes the $\bar{\alpha}_{M_1}$ for the first beam and $\bar{\alpha}_{M_2}$ for the second beam.

$$\bar{\alpha}_{M_1} = \frac{M_1}{m_1 L_1}, \qquad \bar{\alpha}_{M_2} = \frac{M_2}{m_2 L_2}.$$

The solution $G(x, \xi)$ satisfying the differential equation (A1) is Green's function that is looked for. Thus, $G(x, \xi)$ can be found as

$$G(x, \xi) = \frac{1}{2\beta^3}\{\Phi_4(\beta(x-\xi))\mathrm{H}(x-\xi) + \beta G''(0, \xi)\Phi_3(\beta x) + G'''(0, \xi)\Phi_4(\beta x)\}, \quad \text{(A3)}$$

where

$$G''(0, \xi) = \frac{1}{\Delta}\begin{vmatrix} -\frac{1}{\beta}\Phi_2(\beta(L-\xi)) & \frac{1}{\beta}\Phi_2(\beta L) \\ -A\beta\Phi_4(\beta(L-\xi)) - \Phi_1(\beta(L-\xi)) & \Phi_1(\beta L) + A\beta\Phi_4(\beta L) \end{vmatrix},$$

$$G'''(0, \xi) = \frac{1}{\Delta}\begin{vmatrix} \Phi_1(\beta L) & -\frac{1}{\beta}\Phi_2(\beta(L-\xi)) \\ \beta\Phi_4(\beta L) + A\beta^2\Phi_3(\beta L) & -A\beta\Phi_4(\beta(L-\xi)) - \Phi_1(\beta(L-\xi)) \end{vmatrix},$$

$$\Delta = 2[1 + \cosh\beta L\cos\beta L + A\beta(\sinh\beta L\cos\beta L - \sin\beta L\cosh\beta L)].$$

APPENDIX B

With the boundary value problem formulation, the characteristic equation of the system with a single ss, i.e. $n = 1$, can be given as follows [2]:

$$R = \begin{vmatrix}
R_{11} & R_{12} & R_{13} & R_{14} & R_{15} & R_{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
R_{21} & R_{22} & R_{23} & R_{24} & R_{25} & R_{26} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
R_{31} & R_{32} & R_{33} & R_{34} & R_{35} & R_{36} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
R_{41} & R_{42} & R_{43} & R_{44} & R_{45} & R_{46} & 0 & 0 & 0 & 0 & 0 & 0 & R_{4,13} \\
R_{51} & 0 & R_{53} & 0 & 0 & 0 & 0 & R_{58} & 0 & R_{5,10} & R_{5,11} & R_{5,12} & R_{5,13} \\
0 & 0 & 0 & 0 & 0 & 0 & R_{67} & R_{68} & R_{69} & R_{6,10} & R_{6,11} & R_{6,12} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & R_{77} & R_{78} & R_{79} & R_{7,10} & R_{7,11} & R_{7,12} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & R_{87} & R_{88} & R_{89} & R_{8,10} & R_{8,11} & R_{8,12} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & R_{97} & R_{98} & R_{99} & R_{9,10} & R_{9,11} & R_{9,12} & R_{9,13} \\
0 & R_{10,2} & 0 & R_{10,4} & R_{10,5} & R_{10,6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & R_{11,8} & 0 & R_{11,10} & R_{11,11} & R_{11,12} & 0 \\
0 & R_{12,2} & 0 & R_{12,4} & R_{12,5} & R_{12,6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & R_{13,8} & 0 & R_{13,10} & R_{13,11} & R_{13,12} & 0
\end{vmatrix} = 0 \quad \text{(B1)}$$

The non-zero elements of the frequency determinant above are as follows, where $\bar{\beta}$ is given in (6):

$$R_{11} = \sin(\bar{\beta}\eta_1) - \sinh(\bar{\beta}\eta_1), \qquad R_{12} = -\sin(\bar{\beta}\eta_1),$$

$$R_{13} = \cos(\bar{\beta}\eta_1) - \cosh(\bar{\beta}\eta_1), \qquad R_{14} = -\cos(\bar{\beta}\eta_1),$$

$R_{15} = -\sinh(\bar{\beta}\eta_1), \qquad R_{16} = -\cosh(\bar{\beta}\eta_1),$

$R_{21} = \cos(\bar{\beta}\eta_1) - \cosh(\bar{\beta}\eta_1), \qquad R_{22} = -\cos(\bar{\beta}\eta_1),$

$R_{23} = -\sin(\bar{\beta}\eta_1) - \sinh(\bar{\beta}\eta_1), \qquad R_{24} = \sin(\bar{\beta}\eta_1),$

$R_{25} = -\cosh(\bar{\beta}\eta_1), \qquad R_{26} = -\sinh(\bar{\beta}\eta_1),$

$R_{31} = \sin(\bar{\beta}\eta_1) + \sinh(\bar{\beta}\eta_1), \qquad R_{32} = -\sin(\bar{\beta}\eta_1),$

$R_{33} = \cos(\bar{\beta}\eta_1) + \cosh(\bar{\beta}\eta_1), \qquad R_{34} = -\cos(\bar{\beta}\eta_1),$

$R_{35} = \sinh(\bar{\beta}\eta_1), \qquad R_{36} = \cosh(\bar{\beta}\eta_1),$

$R_{41} = \cos(\bar{\beta}\eta_1) + \cosh(\bar{\beta}\eta_1) + (\alpha_{k_1}/\bar{\beta}^3)(\sin(\bar{\beta}\eta_1) - \sinh(\bar{\beta}\eta_1)), \qquad R_{42} = -\cos(\bar{\beta}\eta_1),$

$R_{43} = -\sin(\bar{\beta}\eta_1) + \sinh(\bar{\beta}\eta_1) + (\alpha_{k_1}/\bar{\beta}^3)(\cos(\bar{\beta}\eta_1) - \cosh(\bar{\beta}\eta_1)), \qquad R_{44} = \sin(\bar{\beta}\eta_1),$

$R_{45} = \cosh(\bar{\beta}\eta_1), \qquad R_{46} = \sinh(\bar{\beta}\eta_1), \qquad R_{4,13} = -\alpha_{k_1}/\bar{\beta}^3$

$R_{51} = (\alpha_{k_1}/\alpha_M\bar{\beta}^4)(\sin(\bar{\beta}\eta_1) - \sinh(\bar{\beta}\eta_1)), \qquad R_{53} = (\alpha_{k_1}/\alpha_M\bar{\beta}^4)(\cos(\bar{\beta}\eta_1) - \cosh(\bar{\beta}\eta_1)),$

$R_{58} = (\alpha_{k_1}\alpha_k/\alpha_M\bar{\beta}^4)\sin(\psi\bar{\beta}), \qquad R_{5,10} = (\alpha_{k_1}\alpha_k/\alpha_M\bar{\beta}^4)\cos(\psi\bar{\beta}),$

$R_{5,11} = (\alpha_{k_1}\alpha_k/\alpha_M\bar{\beta}^4)\sinh(\psi\bar{\beta}), \qquad R_{5,12} = (\alpha_{k_1}\alpha_k/\alpha_M\bar{\beta}^4)\cosh(\psi\bar{\beta}),$

$R_{5,13} = 1 - \alpha_{k_1}(\alpha_k + 1)/\alpha_M\bar{\beta}^4,$

$R_{67} = \sin(\psi\bar{\beta}) - \sinh(\psi\bar{\beta}), \qquad R_{68} = -\sin(\psi\bar{\beta}),$

$R_{69} = \cos(\psi\bar{\beta}) - \cosh(\psi\bar{\beta}), \qquad R_{6,10} = -\cos(\psi\bar{\beta}),$

$R_{6,11} = -\sinh(\psi\bar{\beta}), \qquad R_{6,12} = -\cosh(\psi\bar{\beta}),$

$R_{77} = \cos(\psi\bar{\beta}) - \cosh(\psi\bar{\beta}), \qquad R_{78} = -\cos(\psi\bar{\beta}),$

$R_{79} = -\sin(\psi\bar{\beta}) - \sinh(\psi\bar{\beta}), \qquad R_{7,10} = \sin(\psi\bar{\beta}),$

$R_{7,11} = -\cosh(\psi\bar{\beta}), \qquad R_{7,12} = -\sinh(\psi\bar{\beta}),$

$R_{87} = \sin(\psi\bar{\beta}) + \sinh(\psi\bar{\beta}), \qquad R_{88} = -\sin(\psi\bar{\beta}),$

$R_{89} = \cos(\psi\bar{\beta}) + \cosh(\psi\bar{\beta}), \qquad R_{8,10} = -\cos(\psi\bar{\beta}),$

$R_{8,11} = \sinh(\psi\bar{\beta}), \qquad R_{8,12} = \cosh(\psi\bar{\beta}),$

$R_{97} = \cos(\psi\bar{\beta}) + \cosh(\psi\bar{\beta}), \qquad R_{98} = -\cos(\psi\bar{\beta}) + (\alpha_{k_2}/\bar{\delta}^3\bar{\beta}^3)\sin(\psi\bar{\beta}),$

$R_{99} = -\sinh(\psi\bar{\beta}) + \sinh(\psi\bar{\beta}), \qquad R_{9,10} = \sin(\psi\bar{\beta}) + (\alpha_{k_2}/\bar{\delta}^3\bar{\beta}^3)\cos(\psi\bar{\beta}),$

$R_{9,11} = \cosh(\psi\bar{\beta}) + (\alpha_{k_2}/\bar{\delta}^3\bar{\beta}^3)\sinh(\psi\bar{\beta}), \qquad R_{9,12} = \sinh(\psi\bar{\beta}) + (\alpha_{k_2}/\bar{\delta}^3\bar{\beta}^3)\cosh(\psi\bar{\beta}),$

$R_{9,13} = -\alpha_{k_2}/\bar{\delta}^3\bar{\beta}^3,$

$$R_{10,2} = \sin(\bar{\beta}), \qquad R_{10,4} = \cos(\bar{\beta}), \qquad R_{10,5} = -\sinh(\bar{\beta}), \qquad R_{10,6} = -\cosh(\bar{\beta}),$$

$$R_{11,8} = \sin(\bar{\delta}\bar{\beta}), \qquad R_{11,10} = \cos(\bar{\delta}\bar{\beta}), \qquad R_{11,11} = -\sinh(\bar{\delta}\bar{\beta}), \qquad R_{11,12} = -\cosh(\bar{\delta}\bar{\beta}),$$

$$R_{12,2} = -\cos(\bar{\beta}) + \bar{\alpha}_{M_1}\bar{\beta}\sin(\bar{\beta}), \qquad R_{12,4} = \sin(\bar{\beta}) + \bar{\alpha}_{M_1}\bar{\beta}\cos(\bar{\beta}),$$

$$R_{12,5} = \cosh(\bar{\beta}) + \bar{\alpha}_{M_1}\bar{\beta}\sinh(\bar{\beta}), \qquad R_{12,6} = \sinh(\bar{\beta}) + \bar{\alpha}_{M_1}\bar{\beta}\cosh(\bar{\beta}),$$

$$R_{13,8} = -\cos(\bar{\delta}\bar{\beta}) + \bar{\alpha}_{M_2}\bar{\delta}\bar{\beta}\sin(\bar{\delta}\bar{\beta}), \qquad R_{13,10} = \sin(\bar{\delta}\bar{\beta}) + \bar{\alpha}_{M_2}\bar{\delta}\bar{\beta}\cos(\bar{\delta}\bar{\beta}),$$

$$R_{13,11} = \cosh(\bar{\delta}\bar{\beta}) + \bar{\alpha}_{M_2}\bar{\delta}\bar{\beta}\sinh(\bar{\delta}\bar{\beta}), \qquad R_{13,12} = \sinh(\bar{\delta}\bar{\beta}) + \bar{\alpha}_{M_2}\bar{\delta}\bar{\beta}\cosh(\bar{\delta}\bar{\beta}).$$